International Journal of Theoretical Physics, Vol. 36, No. 9, 1997

Moment Problem for Effect Algebras

Miloslav Duchoň,[1] Anatolij Dvurečenskij,[1] and Paolo de Lucia[2]

Received November 13, 1996

We present a solution to the moment problem for effect algebras, concerning mean values of all powers of an observable concentrated on the interval [0, 1] for states from a convex set. We give a solution for particular examples, e.g., for the set of all effect operators. We examine how this problem is related to a so-called E-property. Finally, we give a solution for observables studied in the operational approach to physical theories.

1. INTRODUCTION

The Hausdorff problem is the following: Given a set of real numbers $\{v_n\}_{n=0}^{\infty}$, find a bounded nondecreasing function $u(t)$ on the interval [0, 1] such that for its moments we have

$$v_n = \int_0^1 t^n \, du(t), \qquad n \geq 0$$

Equivalently, find a probability measure μ on the Borel σ-algebra $\mathcal{B}([0, 1])$ such that

$$v_n = \int_{[0,1]} t^n \, d\mu(t), \qquad n \geq 0$$

[1] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, SK-814 73 Bratislava, Slovakia; e-mail: duchon@mau.savba.sk, dvurecen@mau.savba.sk.
[2] Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Complesso Universitario, "Monte S. Angelo", Via Cintia, I-801 26 Naples, Italy; e-mail: delucia@matna2.dma.unima.it.

0020-7748/97/0900-1941$12.50/0

Hausdorff (1921a, b, 1923) showed that the answer is positive iff $\{v_n\}_{n=0}^{\infty}$ is a so-called completely monotone sequence, i.e., for all n, $k \geq 0$,

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} v_{n+j} \geq 0$$

For reviews of these problems see, e.g., Shohat and Tamarkin (1943) and Widder (1946); for operators on Hilbert space some results are given in Riesz and Sg.-Nagy (1955).

In the present paper, we extend this result for observables on effect algebras. These algebras have been recently introduced by Kôpka and Chovanec (1994) (as difference posets), Giuntini and Greuling (1989) (as weak orthoalgebras), and Foulis and Bennett (1993) (as effect algebras). The effect algebras have been widely adopted as physical models generalizing the systems of all effect operators on Hilbert space quantum mechanics (Beltrametti and Bugajski, n.d.).

The solution to the moment problem is given in Section 3 for effects on a convex, order-determining set $\mathcal{S}$ of σ-additive states on a σ-effect algebra; i.e., for affine fuzzy sets on $\mathcal{S}$.

In Section 4 we present a solution, in a particular case for the set of all von Neumann operators of Hilbert space quantum mechanics. In Section 5, we show how this problem is related to sets of states having a so-called E-property in effect algebras. We introduce the class of effect algebras which roughly speaking have the E-property, and we present examples of convex effect algebras.

Finally, in Section 6 we give a solution for observables (called here generalized observables) as affine mappings from a set of states into the set of all probability measures on $\mathcal{B}(R)$. For this operational approach, we give an equivalent solution in an appropriate effect algebra (convex effect algebra) of affine fuzzy sets on a given set of states.

2. EFFECT ALGEBRAS

An *effect algebra* (Foulis and Bennett, 1993; Kôpka and Chovanec, 1994) is a set L with two particular elements 0, 1, and with a partial binary operation $\oplus$: $L \times L \to L$ such that for all $a, b, c \in L$ we have:

(EAi) If $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (commutativity).

(EAii) If $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity).

(EAiii) For any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation).

(EAiv) If $1 \oplus a$ is defined, then $a = 0$ (zero–one law).

If the assumptions of (EAii) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L. As usual, we shall write $L = (L, \oplus, 0, 1)$ for effect algebras.

Let a and b be two elements of an effect algebra L. We say that (i) a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined in L; (ii) a is *less than or equal to* b and write $a \leq b$ iff there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ (in this case we also write $b \geq a$); (iii) b is the *orthocomplement* of a iff b is a (unique) element of L such that $b \perp a$ and $a \oplus b = 1$ and it is written as $a^{\perp}$. If $c = a \oplus b$, we shall write $a = c \ominus b$ and $b = a \ominus a$.

Let $F = \{a_1, \ldots, a_n\}$ be a finite sequence in L. Recursively, we define for $n \geq 3$

$$a_1 \oplus \cdots \oplus a_n := (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n \tag{1}$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in L. From the associativity of $\oplus$ in D-posets we conclude that (1) is correctly defined. By definition we put $a_1 \oplus \cdots \oplus a_n = a_1$ if $n = 1$, and $a_1 \oplus \cdots \oplus a_n = 0$ if $n = 0$. Then for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ and any k with $1 \leq k \leq n$ we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n} \tag{2}$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n) \tag{3}$$

We say that a finite sequence $F = \{a_1, \ldots, a_n\}$ in L is *$\oplus$-orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists in L. In this case we say that F has a *$\oplus$-sum*, $\oplus_{i=1}^{n} a_i$, defined via

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus \cdots \oplus a_n \tag{4}$$

It is clear that two elements a and b of L are orthogonal, i.e., $a \perp b$, iff $\{a, b\}$ is $\oplus$-orthogonal.

An arbitrary system $G = \{a_i\}_{i \in I}$ of not necessarily different elements of L is *$\oplus$-orthogonal* iff, for every finite subset F of I, the system $\{a_i\}_{i \in F}$ is $\oplus$-orthogonal. If $G = \{a_i\}_{i \in I}$ is $\oplus$-orthogonal, so is any $\{a_i\}_{i \in J}$ for any $J \subseteq I$. An $\oplus$-orthogonal system $G = \{a_i\}_{i \in I}$ of L has a *$\oplus$-sum* in L, written as $\oplus_{i \in I} a_i$, iff in L there exists the join

$$\bigoplus_{i \in I} a_i := \bigvee_{F} \bigoplus_{i \in F} a_i \tag{5}$$

where F runs over all finite subsets in I. In this case, we also write $\oplus G := \oplus_{i \in I} a_i$.

It is evident that if $G = \{a_1, \ldots, a_n\}$ is $\oplus$-orthogonal, then the $\oplus$-sums defined by (4) and (5) coincide.

We recall that if $G = \{a_i\}_{i\in I}$ and $a_i = a$ for infinitely many i's, $a = 0$ whenever $\oplus\, G$ exists in L. Indeed, let $a_0 = \oplus\, G$; then

$$a_0 = a_{i_0} \oplus \bigoplus_{i\in I\setminus\{i_0\}} a_i = a \oplus a_0$$

which gives a $= 0$. On the other hand, if G is only $\oplus$-orthogonal, then a is not necessarily 0.[3]

We say that an effect algebra L is a *σ-effect algebra* (*complete effect algebra*) if $\oplus_{i\in I}\, a_i$ belongs to L for any countable (arbitrary) system $\{a_i\colon i \in I\}$ of $\oplus$-orthogonal elements from L.

Two prototypes of effect algebras are the following two examples.

Example 2.1. The set $\mathscr{E}(H)$ of all Hermitian operators A on H such that $O \le A \le I$, where I is the identity operator on H, is an effect algebra; a partial ordering $\le$ is defined via $A \le B$ iff $(Ax, x) \le (Bx, x)$, $x \in H$, and $C = A \oplus B$, $C \in \mathscr{E}(H)$, iff $(Ax, x) + (Bx, x) = (Cx, x)$, $x \in H$. A system $\{A_i\colon i \in I\}$ from $\mathscr{E}(H)$ is $\oplus$-orthogonal iff $\Sigma_{i\in I} A_i \in \mathscr{E}(H)$ (where the convergence in the summation is, e.g., weak or strong) and then $\oplus_{i\in I} A_i = \Sigma_{i\in I} A_i$. In addition, $\mathscr{E}(H)$ is a complete effect algebra which is not a lattice.

This example, $\mathscr{E}(H)$, plays an important role in the unsharp measurement in quantum mechanics.

Example 2.2. Let the closed interval [0, 1] be ordered in the natural way, and, for two numbers $a, b \in [0, 1]$, we define $a \oplus b$ iff $a + b \le 1$ and we put then $a \oplus b = a + b$. Then [0, 1] is a totally ordered, distributive lattice in that any $\oplus$-orthogonal system has the sum in it. We recall that $\{a_t\}$ is $\oplus$-orthogonal iff $\{a_t\}$ is summable and $\Sigma_t\, a_t \le 1$; then $\oplus_s\, a_t = \Sigma_t\, a_t$.

A real-valued mapping s on an effect algebra L is said to be a *state* if (i) $s(1) = 1$, and (ii) $s(a \oplus b) = s(a) + s(b)$, $a, b \in L$. It is clear that $m(0) = 0$.

If for a state $s\colon L \to [0, 1]$ we have

$$s(\bigoplus_{i\in I} a_i) = \sum_{i\in I} s(a_i) \tag{6}$$

whenever $\oplus_{i\in I} a_i$ exists in L, then s is said to be a *σ-additive* or *completely additive state* if (6) holds for any countable or any index set I, respectively.

[3] For example, let $\{n, \dot{n}\colon n \ge 0\}$, where $0 < 1 < 2 < \cdots < n < n + 1 < \cdots < (n \dot{+} 1) < \dot{n} < \cdots < \dot{2} < \dot{1} < \dot{0}$. We define $n^\perp := \dot{n}$, $(\dot{n})^\perp := n$ for any $n \ge 0$, and we put $m \oplus n := m + n$ for all $m, n \ge 0$, and $n \oplus \dot{m} := (m \dot{-} n)$ if $n \le m$. Then $L = (L, \oplus, 0, \dot{0})$ is an effect algebra. If $G = \{a_i\}_{i=1}^{\infty}$, where $a_i = 1$, then G is $\oplus$-orthogonal, but $\oplus\, G$ does not exist in L.

A nonvoid system of states $\mathcal{S}$ on L is said to be *order determining* if, for $a, b \in L$, $a \leq b$ iff $s(a) \leq s(b)$ for any $s \in \mathcal{S}$. We denote by $Con(\mathcal{S})$ and $Con_\sigma(\mathcal{S})$ the convex hull and the σ-convex hull of $\mathcal{S}$, respectively. It is clear that $\mathcal{S}$ is order determining iff $Con(\mathcal{S})$ is so, or, equivalently, iff $Con_\sigma(\mathcal{S})$ is order determining.

We say that two effect algebras L and P are *isomorphic* if there exists a one-to-one mapping ϕ from L onto P such that $\phi(1) = 1$, and $a \oplus b \in L$ iff $\phi(a) \oplus \phi(b) \in P$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. If L and P are σ-effect algebras (complete effect algebras), we assume in addition for any isomorphism ϕ that it preserves all sums of all countable (arbitrary) $\oplus$-orthogonal systems in L.

For σ-effect algebras with an order-determining system of σ-additive states we have the following representation (Dvurečenskij, 1993) via fuzzy set [in Dvurečenskij (1993) there are also representations for effect algebras or complete algebras]. We recall that if Ω is a nonempty set, by 0_Ω and 1_Ω we understand the constant functions $0_\Omega(\omega) = 0$ and $1_\Omega(\omega) = 1$ for each $\omega \in \Omega$. For two fuzzy sets f and g of Ω we write $f \leq g$ iff $f(\omega) \leq g(\omega)$ for any $\omega \in \Omega$. Similarly $f + g$ and $f - g$ are defined pointwise.

Theorem 2.3. Let a system of fuzzy sets $L \subseteq [0, 1]^\Omega$, $\Omega \neq \emptyset$, satisfy the following conditions:

(i) $1_\Omega \in L$.
(ii) $1_\Omega - f \in L$ whenever $f \in L$.
(iii) if for a sequence $\{f_i\}$ from L with $\Sigma_{i=1}^n f_i \in L$ for any $n \geq 1$, then $\Sigma_{i=1}^\infty f_i \in L$.

Then $L = (L, \oplus, 0_\Omega, 1_\Omega)$, where $f \oplus g$ is defined if and only if $f + g \leq 1_\Omega$ ($f, g \in L$) and we put $f \oplus g = f + g$, is a σ-effect algebra. In addition, the system $\mathcal{S} = \{s_\omega: \omega \in \Omega\}$, where $s_\omega: L \to [0, 1]$ is defined via $s_\omega(f) := f(\omega)$, $f \in L$, is an order-determining system of σ-additive states.

Conversely, let $L = (L, \oplus, 0, 1)$ be an arbitrary σ-effect algebra with an order-determining system of a σ-additive states $\mathcal{S}$. Then L is isomorphic with the system of fuzzy sets $\overline{L} \subseteq [0, 1]^s$, where $\overline{L} = \{\overline{a} \in [0, 1]^s: \overline{a}(s) := s(a), s \in \mathcal{S}, a \in L\}$, and $\overline{L}$ satisfies the conditions (i)–(iii).

3. MOMENT PROBLEM AND OBSERVABLES

Let L be a σ-effect algebra. By an observable of L we mean any mapping $x: \mathcal{B}(R) \to L$ such that:

(i) $x(R) = 1$.

(ii) $x(\cup_{i=1}^{\infty} E_i) = \oplus_{i=1}^{\infty} x(E_i)$ whenever $E_i \cap E_j = \emptyset$ for $i \neq j$, $E_i \in \mathcal{B}(\mathcal{R})$ for $i \geq 1$.

An observable x is *bounded* if there is a bounded set C such that $x(C) = 1$.

If x is an observable of a σ-effect algebra L, then there is the least closed subset $\sigma(x)$ (called the *spectrum* of x) such that $x(\sigma(x)) = 1$. Indeed, let O_1 and O_2 be two open sets in R such that $x(O_1) = 0 = x(O_2)$. Then

$$\begin{aligned} x(O_1 \cup O_2) &= x((O_1 \setminus O_2) \cup (O_1 \cap O_2) \cup (O_2 \setminus O_1)) \\ &= x(O_1 \setminus O_2) \oplus x(O_1 \cup O_2) \oplus x(O_2 \setminus O_1) = 0 \end{aligned}$$

Hence $x(\cap_{n=1}^{\infty} O_n) = 0$ for any sequence of open subsets $\{O_n\}$ with $x(O_n) = 0$ $(n \geq 1)$. Hence, $\sigma(x) = \cap\{C: x(C) = 1, C \text{ is closed}\}$ satisfies $x(\sigma(x)) = 1$, which is possible because the topology of R satisfies the second countability axiom.

If f is a Borel function on R, then $f \circ x: E \mapsto x(f^{-1}(E))$, $E \in \mathcal{B}(R)$, is an observable of L, too. In particular, if $f_n(t) = t^n$, $t \in R$ $(n \geq 1)$, we denote $x^n := f_n \circ x$.

If x is an observable and s is a σ-additive state on L, then $s_x: \mathcal{B}(R) \to [0, 1]$ defined via

$$s_x(E) = s(x(E)), \qquad E \in \mathcal{B}(R)$$

is a probability measure on $\mathcal{B}(R)$ and we denote by

$$s(x) := \int_R t \, ds_x(t)$$

the *mean value* of x in s whenever the right-hand side of the former equation exists and is finite. Hence,

$$s(x^n) = \int_R t \, ds_{x^n}(t) = \int_R t^n \, ds_x(t)$$

In what follows, we shall study observables concentrated on the interval $[0, 1]$, i.e., observables x with $x([0, 1]) = 1$. Denote $\mathcal{B}_1 := \mathcal{B}(R) \cap [0, 1]$.

Let $\mathcal{S} \neq \emptyset$ be a convex system of σ-additive states on L. A mapping $f: \mathcal{B} \times \mathcal{B}(R) \to [0, 1]$ such that

(i) given $s \in \mathcal{S}$, $f(s, \cdot)$ is a finitely additive measure on $\mathcal{B}(R)$;
(ii) for any $M \in \mathcal{B}(R)$, $f(\lambda s_1 + (1 - \lambda) s_2, M) = \lambda f(s_1, M) + (1 - \lambda) f(s_2, M)$ whenever $\lambda \in [0, 1]$ and $s_1, s_2 \in \mathcal{S}$;

is said to be the *effect function* on $\mathcal{S}$.

We say that a convex system $\mathcal{S}$ of σ-additive states on L has the *E-property* on L (E for "existence") if, given an effect function f on $\mathcal{S}$, for any $M \in \mathcal{B}(R)$, there exists an element $x(M) \in L$ such that

$$f(s, M) = s(x(M)), \qquad s \in \mathcal{S}$$

Let $a_n \in [0, 1]^{\mathcal{S}}$ for $n = 0, 1, \ldots$. We say that a sequence of functions $\{a_n\}_{n=0}^{\infty}$ from $[0, 1]^{\mathcal{S}}$ is *completely monotone* if, for any $n, k \geq 0, 1, \ldots$ and any $s \in \mathcal{S}$, we have

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n+j}(s) \geq 0$$

Lemma 3.1. Let x be an observable on a σ-effect algebra L, which is concentrated on $[0, 1]$, and let $\mathcal{S}$ be a nonempty set of a σ-additive states on L. Define

$$a_n(s) = \int_{[0,1]} t^n \, ds_x(t), \qquad s \in \mathcal{S}, \quad n \geq 0 \tag{7}$$

Then $\{a_n\}_{n=0}^{\infty}$ is a completely monotone sequence of functions from $[0, 1]^{\mathcal{S}}$.

Proof. It follows from the simple observation

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n+j}(s) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \int_{[0,1]} t^{n+j} \, ds_x(t)$$

$$= \int_{[0,1]} t^n (1 - t)^k \, ds_x(t) \geq 0 \quad \text{QED}$$

Theorem 3.2. Let L be a σ-effect algebra and let $\mathcal{S}$ be a convex, order-determining system of σ-additive states on L having the E-property on L. Let $\{a_n\}_{n=0}^{\infty}$ be a completely monotone sequence of functions from $[0, 1]^{\mathcal{S}}$ with $a_0(s) = 1$, $s \in \mathcal{S}$, and, for any n, let

$$a_n(\lambda s_1 + (1 - \lambda)s_2) = \lambda a_n(s_1) + (1 - \lambda)a_n(s_2)$$

for any $\lambda \in [0, 1]$ and all $s_1, s_2 \in \mathcal{S}$. Then there is a unique observable x and L concentrated on $[0, 1]$ such that (7) holds.

Proof. Let $s \in \mathcal{S}$ be a fixed σ-additive state. Since the sequence of real numbers $\{a_n(s)\}_{n=0}^{\infty}$ is completely monotone, according to the classical result of Hausdorff (1923), there is a unique probability measure p_s on $\mathcal{B}_1$ such that

$$a_n(s) = \int_{[0,1]} t^n \, dp_s(t), \qquad n \geq 0 \tag{8}$$

A function $f: \mathcal{S} \times \mathcal{B}(R) \to [0, 1]$ defined via

$$f(s, M) = p_s(M \cap [0, 1]), \qquad M \in \mathcal{B}(R)$$

is an effect function on $\mathcal{S}$. Indeed,

$$1 = a_0(s) = \int_{[0,1]} t^0 \, dp_s(t) = p_s([0, 1])$$

and other properties of effect functions are evident.

The E-property implies that, for any $M \in \mathcal{B}(R)$, there is an element $x(M) \in L$ such that

$$p_s(M \cap [0, 1]) = s(x(M)), \qquad s \in \mathcal{S} \tag{9}$$

Since $\mathcal{S}$ is an order-determining system of σ-additive states, $x(M)$ is determined uniquely by (9).

We assert that the mapping $x: M \mapsto x(M)$ is an observable on L. Indeed, it satisfies the property (i) of observables. Let now $M = \cup_{i=1}^{\infty} M_i$, where $M_i \cap M_j = \emptyset$ for $i \neq j$, and $M_i \in \mathcal{B}(R)$. Put $F_n = \cup_{i=1}^{n} M_i$ for any $n \geq 1$. Then $x(F_n) = \oplus_{i=1}^{n} x(M_i)$, and

$$s(x(M) \ominus x(F_n)) = s(x(M \setminus F_n)) = p_s((M \setminus F_n) \cap [0, 1]) \searrow 0$$

since any p_s is σ-additive on $\mathcal{B}_1$. Hence, for any $s \in \mathcal{S}$,

$$s(x(M)) = s(\bigoplus_{i=1}^{\infty} x(M_i))$$

so that $x(M) = \oplus_{i=1}^{\infty} x(M_i)$. In addition, $x([0, 1]) = 1$ and x is the unique observable in question. QED

4. MOMENT PROBLEM FOR $\mathcal{E}(H)$

Let H be a real or complex Hilbert space. Denote by $\mathcal{E}(H)$ the set of all effect operators on H (see Example 2.1). If now T is a von Neumann operator on H, i.e., T is a Hermitian positive-trace operator on H with $\mathrm{tr}(T) = 1$, then the mapping $m_T: \mathcal{E}(H) \to [0, 1]$ defined via

$$m_T(A) := \mathrm{tr}(TA), \qquad A \in \mathcal{E}(H) \tag{10}$$

is a completely additive state on $\mathcal{E}(H)$. In particular, if ϕ is a unit vector in H, then the mapping

$$m_\phi(A) := (A\phi, \phi), \qquad A \in \mathcal{E}(H) \tag{11}$$

is a completely additive state on $\mathcal{E}(H)$. The Gleason theorem on $\mathcal{E}(H)$ says (Dvurecenskij, 1993) that if $\dim H \geq 3$, then any completely additive measure on $\mathcal{E}(H)$ is of the form (10).

Denote by $\mathrm{Tr}(H)$ and $\mathrm{Tr}_1(H)$ the sets of all Hermitian trace operators and von Neumann operators on H, respectively.

We recall that the space $\mathscr{E}(H)$ is important for unsharp measurement in quantum mechanics (Busch *et al.*, 1991) because there is an intimate connection between observables on $\mathscr{E}(H)$ and so-called POV-measures.

We recall that a *POV-measure* is a mapping E: $\mathscr{B}(R) \to \mathscr{E}(H)$ such that:

(i) $E(R) = I$;
(ii) $E(\cup_{i=1}^{\infty} M_i) = \Sigma_{i=1}^{\infty} E(M_i)$ for any sequence of disjoint sets $\{M_i\}$ in $\mathscr{B}(R)$;

where the sum converges in the weak operator topology. We note that observables on $\mathscr{E}(H)$ and POV-measures are the same things.

Let ϕ be a unit vector in H and let m_ϕ be a state on $\mathscr{E}(H)$ defined by (11). Take an observable $E(=$ POV-measure) on $\mathscr{E}(H)$ concentrated on [0, 1]. Then the expression $\int_{[0,1]} t^n \, dm_{\phi_E}(t)$, $n \geq 0$, defines a Hermitian operator E_n from $\mathscr{E}(H)$, called the nth *moment operator of E*, such that

$$(E_n\phi, \phi) = \int_{[0,1]} t^n \, dm_{\phi_E}(t) \tag{12}$$

and the sequence $\{E_n\}_{n=0}^{\infty}$ is completely monotone, i.e., for all integers n, $k \geq 0$,

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} e_{n+j} \geq O \tag{13}$$

where O is the null operator on H.

We now show that the converse statement holds for completely monotone Hermitian operators, i.e., any such sequence from $\mathscr{E}(H)$ is a sequence of moment operators of some POV-measure.

Theorem 4.1. Let $L = \mathscr{E}(H)$ and $\mathscr{S} = \mathscr{S}(H) := \{m_T: T \in \mathrm{Tr}_1(H)\}$. Then $\mathscr{S}$ has the E-property on $\mathscr{E}(H)$.

Proof. It is evident that $\mathscr{S}$ is a convex, nonempty set of completely additive states on $\mathscr{E}(H)$. Let now f: $\mathscr{S} \times \mathscr{B}(R) \to [0, 1]$ be any effect function on $\mathscr{S}$, and define, for any $M \in \mathscr{B}(R)$, an affine functional f_M: $\mathrm{Tr}_1(H) \to [0, 1]$ via $f_M(T) := f(m_T, M)$, $T \in \mathrm{Tr}_1(H)$. This functional can be uniquely extended to a bounded linear functional $\tilde{f}_M$ on the set $\mathrm{Tr}(H)$ with the norm $\|\tilde{f}_M\| \leq 1$. According to a representation theorem of bounded real-valued linear functionals on $\mathrm{Tr}(H)$, Theorem VI.26.b in Reed and Simon (1972), there is a unique Hermitian operator $E(M) \in \mathscr{E}(H)$ such that

$$\tilde{f}_M(T) = \mathrm{tr}(TE(M)), \qquad T \in \mathrm{Tr}(H)$$

This implies

$$f_M(T) = f(m_T, M) = m_T(E(M))$$

for any $m_T \in \mathcal{S}$.

Theorem 4.2. Let $\{E_n\}_{n=0}^{\infty}$ with $E_0 = I$ be a completely monotone sequence of effect operators on $\mathcal{E}(H)$. Then there is a unique POV-measure E concentrated on the interval [0, 1] such that E_n is the nth moment operator of E.

Proof. For any unit vector $\phi \in H$, the mapping T_ϕ: $H \to H$ defined as $T_\phi(\psi) := (\psi, \phi)\,\phi$, $\psi \in H$, is a von Neumann operator on H, and we have that

$$m_{T_\phi}(A) = (A\phi, \phi), \qquad A \in \mathcal{E}(H)$$

i.e., $m_{T_\phi} = m_\phi$. It is clear that the set $\mathcal{S}_0 := \{m_\phi: \|\phi\| = 1\}$ is an order-determining system of completely additive states, and $Con_\sigma(\mathcal{S}_0) = \mathcal{S}(H)$, where $\mathcal{S}(H)$ is from Theorem 4.1. Since $T = \Sigma_i \lambda_i T_{\phi_i}$, where $\lambda_i > 0$, $\Sigma_i \lambda_i = 1$, and $\phi_i \perp \phi_j$ for $i \neq j$, we have

$$m_T(E_n) = \text{tr}(TE_n) = \sum_i \lambda_i(E_n\phi_i, \phi_i)$$

In view of complete monotonicity of the sequence of Hermitian operators $\{E_n\}_{n=0}^{\infty}$, $\{\tilde{E}_n\}_{n=0}^{\infty}$, where $\tilde{E}_n \in [0, 1]^{\mathcal{S}}$ is defined via

$$\tilde{E}_n(m_T) := \text{tr}(TE_n), \qquad T \in \text{Tr}(H), \quad n \geq 0$$

is a completely monotome sequence of functions from $[0, 1]^{\mathcal{S}}$. Since $\mathcal{S}$ has the E-property, applying Theorem 3.2 to $\{\tilde{E}_n\}_{n=0}^{\infty}$, we obtain a unique observable (= POV-measure) E: $\mathcal{B}(R) \to \mathcal{E}(H)$, concentrated on [0, 1] such that

$$\tilde{E}_n(m_T) = \int_{[0,1]} t^n \, dm_{T_E}(t), \qquad n \geq 0$$

In particular, we have

$$(E_n\phi, \phi) = \int_{[0,1]} t^n \, d(E(t)\phi, \phi) = \int_{[0,1]} t^n \, dm_{\phi_E}(t), \qquad n \geq 0 \quad \text{QED}$$

Let now $\mathcal{L}(H)$ be the system of all orthogonal projections on H. Hence $\mathcal{L}(H) \subset \mathcal{E}(H)$, and $\mathcal{L}(H)$ can be viewed as a complete orthomodular lattice with respect to the usual ordering of operators. In addition, $P \oplus Q = P \vee Q = P + Q$ whenever $P + Q \leq I$, and $\mathcal{L}(H)$ is a complete effect algebra, too. Any observable x: $\mathcal{B}(R) \to \mathcal{L}(H)$ can be understood as a special POV-measure, a so-called *PV-measure*.

A POV-measure E is a PV-measure iff $E(M \cap N) = P(M)P(N)$ for any $M, N \in \mathcal{B}(R)$, or, equivalently, if $E(M)^2 = E(M)$ for any Borel subsets M.

We recall that if E is a PV-measure, then for its nth moment operator

$$E_n = \int t^n \, dE(t), \qquad n \geq 0 \tag{14}$$

we have

$$E_n = E_1^n \tag{15}$$

Theorem 4.3. Let $\{E_n\}_{n=0}^{\infty}$ be a sequence of effect operators from $\mathscr{E}(H)$ with $E_0 = I$. Then there is a unique PV-measure E concentrated on [0, 1] such that E_n is the nth moment operator of E if and only if $\{E_n\}_{n=0}^{\infty}$ is a completely monotone sequence and

$$E_2 \leq E_1^2$$

Proof. If E is a PV-measure on $\mathscr{E}(H)$, then the assertion of the theorem is true. Suppose the converse. Then from Theorem 4.2 we have the existence of a POV-measure E concentrated on the interval [0, 1] such that E_n is its nth operator moment. Using the Kadison result (Riesz and Sz-Nagy, 1955, p. 448; Kadison, 1952), i.e., E is a PV-measure iff $E_1^2 \leq E_2$, we see that E is a PV-measure. QED

Remark 4.4. It is worth noting that Riesz and Sz.-Nagy (1955, p. 445) proved also an analogue of the moment problem which can be formulated as follows: If $\{E_n\}_{n=0}^{\infty}$ is a sequence of Hermitian operators on H with $E_0 = I$ such that $a_0E_0 + a_1E_1 + \cdots + a_nE_n \geq O$ whenever $a_0 + a_1t + \cdots + a_nt^n \geq 0$ for any $t \in [-M, M]$ $(a_0, a_1, \ldots, a_n \in R)$, then there is a POV-measure E concentrated on $[-M, M]$ such that E_n is its nth moment operator.

5. EFFECT ALGEBRAS AND E-PROPERTY

In the present section, we give examples of σ-effect algebras having a non-empty, convex system of order-determining σ-additive states for which the E-property holds, and also ones for which it fails.

From Theorem 4.1 we know that for the most important example of effect algebras for quantum physics, $\mathscr{E}(H)$, the set of all states corresponding to von Neumann operators via (10) has the E-property in $\mathscr{E}(H)$.

Example 5.1. Let $L = [0, 1]$ be ordered in the natural way (see Example 2.2). Then the set of all states $\mathcal{S} = \{s_0\}$, where $s_0(t) = t$, $t \in [0, 1]$, is order determining. This state is also completely additive. Then $\mathcal{S}$ has the E-property in [0, 1].

Proof. Let $f: \mathcal{S} \times \mathcal{B}(R) \to [0, 1]$ be an effect function. For any $M \in \mathcal{B}(R)$ we put $x(M) = f(s_0, M)$; then $s_0(x(M)) = x(M) = f(s_0, M)$. QED

Example 5.2. Let $[0, 1]^2 = [0, 1] \times [0, 1]$ be the product of two copies of [0, 1]. Then $[0, 1]^2$ is a complete effect algebra, where $(u_1, v_1) \oplus (u_2, v_2)$ is defined iff $u_1 \oplus u_2$ and $v_1 \oplus v_2$ are defined in [0, 1]; then $(u_1, v_1) \oplus (u_2, v_2) := (u_1 \oplus u_2, v_1 \oplus v_2)$. The space of all states on $[0, 1]^2$ is the set $\mathcal{S} = \{s_\alpha: \alpha \in [0, 1]\}$, where $s_\alpha(u, v) := \alpha u + (1 - \alpha)v$, $(u, v) \in [0, 1]^2$, which is order determining. Any state is completely additive, and $\mathcal{S}$ has the E-property in $[0, 1]^2$.

Proof. Let $f: \mathcal{S} \times \mathcal{B}(R) \to [0, 1]$ be an effect function. Given $M \in \mathcal{B}(R)$, we define

$$x(M) := (f(s_0, M), f(s_1, M)) \in [0, 1]^2$$

Then

$$\begin{aligned} s_\alpha(x(M)) &= \alpha f(s_0, M) + (1 - \alpha)f(s_1, M) \\ &= f(\alpha s_0 + (1 - \alpha)s_1, M) = f(s_\alpha, M) \quad \text{QED} \end{aligned}$$

Example 5.3. Let $L_n = [0, 1]^n$ be the product of n copies of [0, 1] ($n \geq 1$). Then, similarly as in Example 5.2, L_n is a complete effect algebra, and the space of all states $\mathcal{S} = \mathcal{S}(L_n)$ on L_n coincides with the set $\{s_{\alpha_1 \cdots \alpha_n}: \alpha_i \geq 0, i = 1, \ldots, n, \Sigma_{i=1}^n \alpha_n = 1\}$, where $s_{\alpha_1 \cdots \alpha_n}(u_1, \ldots, u_n) := \Sigma_{i=1}^n \alpha_i u_i$. Then $\mathcal{S}$ is order determining and has the E-property in L_n.

Proof. Let f be an effect function. Define $x(M) = (u_1, \ldots, u_n)$, where $u_i = f(s_i, M)$, and s_i is an $s_{\beta_1 \cdots \beta_n}$ having $\beta_i = 1$. Then $s_{\alpha_1 \cdots \alpha_n}(x(M)) = f(s_{\alpha_1 \cdots \alpha_n}, M)$, $M \in \mathcal{B}(R)$. QED

Example 5.4. Let $L = \mathcal{L}(H)$ be the system of all orthogonal projections on a Hilbert space H. According to the famous Gleason theorem (Dvurečensky, 1993) if $\dim H \geq 3$, any completely additive state on it is given by formula (10) restricted to $\mathcal{L}(H)$. Then $\mathcal{S}(H) = \{m_T: T \in \mathrm{Tr}_1(H)\}$ does have, in general, the E-property in $\mathcal{L}(H)$.

Proof. Let A be an effect operator which is no orthogonal projection, and let E_A be its spectral measure. Define an effect function f on $\mathcal{S}(H)$ via

$$f(m_T, M) := \mathrm{tr}(TE_A(M)), \qquad T \in \mathrm{Tr}_1(H)$$

Then this effect function has no solution in $\mathcal{L}(H)$. QED

Example 5.5. Let L_Q: $[0, 1] \cap Q$, where Q is the set of all rational numbers. Then L_Q is an effect algebra which is not a σ-algebra. It possesses a unique state s_0, namely $s_0(r) := r$, $r \in L_Q$, which is a completely additive state. For the convex singleton $\{s_0\}$ the E-property fails in L_Q.

Comparing Example 5.4 and Theorem 4.1, we see that for affinely isomorphic sets of completely additive states on $\mathscr{E}(H)$ and $\mathscr{L}(H)$, respectively, we have completely different situations with validity of the E-property on them.

Let $\mathscr{S}$ be a nonvoid, convex set. An *effect* on $\mathscr{S}$ is any affine mapping f: $\mathscr{S} \to [0, 1]$. Denote by $\mathscr{E}(\mathscr{S})$ the set of all effects on $\mathscr{S}$. Then $\mathscr{E}(\mathscr{S})$ is a complete effect algebra under the usual addition when $f \oplus g$ is defined in $\mathscr{E}(\mathscr{S})$ if and only if $f(s) + g(s) \leq 1$ for any $s \in \mathscr{S}$ and $(f \oplus g)(s) = f(s) + g(s)$, with the least and greatest elements $0_{\mathscr{S}}$ and $1_{\mathscr{S}}$, respectively, and with a convex set of completely additive states $\overline{\mathscr{S}} := \{\bar{s}: s \in \mathscr{S}\}$, where $\bar{s}(f) = f(s)$, $f \in \mathscr{E}(\mathscr{S})$. This set of states is order determining on $\mathscr{E}(\mathscr{S})$ [compare with Theorem 2.3 and Dvurečenskij (n.d.)].

Let now L be a σ-effect algebra with a convex, nonvoid, order-determining set of σ-additive states $\mathscr{S}$. Then L can be naturally embedded into $\mathscr{E}(\mathscr{S})$: For any $a \in L$ we assign $\bar{a}$: $\mathscr{S} \to [0, 1]$ as $\bar{a}(s) := s(a)$, $s \in \mathscr{S}$. Motivated by this, a σ-effect algebra L with a convex, order-determining system of σ-additive states $\mathscr{S} \neq \emptyset$ is said to be a *convex effect algebra* if L can be embedded surjectively into $\mathscr{E}(\mathscr{S})$.[4]

It is clear that $\mathscr{E}(\mathscr{S})$ with $\overline{\mathscr{S}}$ is always a convex effect algebra.

The notion of convex effect algebras gives us the following by-product.

Theorem 5.6. Let L be an effect algebra (σ-effect algebra, complete effect algebra, respectively) with a nonvoid, convex set of states (σ-additive states, completely additive states) $\mathscr{S}$ which is order determining. Then there is a complete effect algebra E with a convex system of completely additive states affinely isomorphic with $\mathscr{S}$ such that L can be embedded into E, and any state on L can be extended into a completely additive state on E.

Proof. Using a fuzzy set representation [see Theorem 2.1 and Dvurečenskij (n.d.)], any effect algebra (σ-effect algebra, complete effect algebra) can be represented by fuzzy sets on $\mathscr{S}$, more precisely by effects on $\mathscr{S}$. If we put $E := \mathscr{E}(\mathscr{S})$, the set of all effects on $\mathscr{S}$, we give the complete effect algebra in question. QED

For example, $\mathscr{L}(H)$ can be embedded into $\mathscr{E}(H)$ and L_Q from Example 5.5 into [0, 1].

[4] A similar notion related to convex effect algebras, *consistent effect algebras*, has been recently introduced by Beltrametti and Bugajski (n.d.).

Theorem 5.7. Let L with a convex, nonvoid, order-determining set of σ-additive states $\mathcal{S} \neq \emptyset$ be a convex effect algebra. Then $\mathcal{S}$ has the E-property in L.

Proof. Let $f: \mathcal{S} \times \mathcal{B}(R) \to [0, 1]$ be any effect function on $\mathcal{S}$. Then the function $f_M(s) := f(s, M)$, $s \in \mathcal{S}$, is an effect on $\mathcal{S}$ for any $M \in \mathcal{B}(R)$. Hence there is an element $x(M) \in L$ such that $f_M = \overline{x(M)} \in \mathcal{E}(\mathcal{S})$, and we have

$$s(x(M)) = f_M(s) = f(s, M)$$

for any $s \in \mathcal{S}$. QED

Examples $\mathcal{E}(H)$ with $\mathcal{S}(H)$ (Theorem 4.1) and Examples 5.1–5.4 are examples of convex effect algebras, while Example 5.4 is no convex effect algebra.

Convex effect algebras give many examples of σ-effect algebras starting with a nonvoid convex abstract set $\mathcal{S}$. Roughly speaking, convex effect algebras are exactly effect algebras with a nonempty, convex system of states being order determining and having the E-property. In addition, those effect algebras have a fuzzy set-representation by effects.

Theorem 5.8. Let L_i with $\mathcal{S}_i$ be a convex effect algebra ($i = 1, \ldots, n$). Then $L = L_1 \times \cdots \times L_n$, where $\oplus$ is defined pointwisely (see Examples 5.3–5.4), is a convex effect algebra with a convex system of σ-additive states

$$\mathcal{S} = \{(\alpha_1 s_1, \ldots, \alpha_n s_n): \alpha \geq 0, s_i \in \mathcal{S}_i, \sum_{i=1}^{n} \alpha_i = 1\}$$

where $(\alpha_1 s_1, \ldots, \alpha_n s_n)(u_1, \ldots, u_n) := \Sigma_{i=1}^{n} \alpha_i s_i(u_i)$.

Proof. First of all, $\mathcal{S}$ is a convex, order-determining set of σ-additive states on L. Let now $f: \mathcal{S} \to [0, 1]$ be an effect on $\mathcal{S}$. Define $f_i: \mathcal{S}_i \to [0, 1]$ as follows:

$$f_i(s_i) := f((0s_1, \ldots, 1s_i, \ldots, 0s_n)), \qquad s_i \in \mathcal{S}_i$$

Then f_i is an effect on $\mathcal{S}_i$. Since L_i is convex, there is an element $u_i \in L_i$ such that $f_i(s) = \overline{u_i}(s) = s(u_i)$ for any $s \in \mathcal{S}_i$ and any $i = 1, \ldots, n$. Then, for the element $u = (u_1, \ldots, u_n)$, we have

$$\begin{aligned}\overline{(u_1, \ldots, u_n)}((\alpha_1 s_1, \ldots, \alpha_n s_n)) &= (\alpha_1 s_1, \ldots, \alpha_n s_n)((u_1, \ldots, u_n)) \\ &= \sum_{i=1}^{n} \alpha_i s_i(u_i) = \sum_{i=1}^{n} \alpha_i f_i(s_i) \\ &= \sum_{i=1}^{n} \alpha_i f((0s_1, \ldots, 1s_i, \ldots, 0s_n)) \\ &= f((\alpha_1 s_1, \ldots, \alpha_n s_n)) \quad \text{QED}\end{aligned}$$

Theorem 5.9. Let L_i be a σ-effect algebra with a convex order-determining system S_i of σ-additive states on L_i, $i = 1, \ldots, n$. Define the product effect algebra $L = L_1 \times \cdots \times L_n$ (see Theorem 5.8). If any $\mathcal{S}_i$ has the E-property in L_i, so has $\mathcal{S}$ from Theorem 5.8 in L.

Proof. It follows the same ideas as the proof of Theorem 5.8. Let f: $\mathcal{S} \times \mathcal{B}(R) \to [0, 1]$ be an effect function. Then f_i: $\mathcal{S}_i \times \mathcal{B}(R) \to [0, 1]$ defined via

$$f_i(s_i, M) := f((0s_1, \ldots, 1s_i, \ldots, 0s_n), M), \qquad s_i \in \mathcal{S}_i$$

is an effect function on $\mathcal{S}_i$. Then there is an element $x_i(M) \in L_i$ such that $f_i(s, M) = s(x_i(M))$ for any $s \in \mathcal{S}_i$, M $\mathcal{B}(R)$. Define $x(M) = (x_1(M), \ldots, x_n(M))$. Then

$$\begin{aligned}(\alpha_1 s_1, \ldots, \alpha_n s_n)(x(M)) &= \sum_{i=1}^{n} \alpha_i s_i(x_i(M)) \\ &= \sum_{i=1}^{n} \alpha_i f_i(s_i, M) \\ &= \sum_{i=1}^{n} \alpha_i f((0s_1, \ldots, 1s_i, \ldots, 0s_n), M) \\ &= f((\alpha_1 s_1, \ldots, \alpha_n s_n), M) \quad \text{QED}\end{aligned}$$

6. MOMENT PROBLEMS FOR GENERALIZED OBSERVABLES

The natural notion of observable as a σ-homomorphism from $\mathcal{B}(R)$ into a σ-effect algebra is generalized as follows in quantum physics. Let $\mathcal{S}$ be a set representing, in a given theoretical model, the set of all σ-additive states of the physical system under consideration . It is assumed that $\mathcal{S}$ is nonvoid and convex. Let $\mathcal{M}(R)$ denote the set of all probability measures on $\mathcal{B}(R)$.

By a *generalized observable* (to distinguish it from observables as σ-homomorphisms) we mean any affine mapping B: $\mathcal{S} \to \mathcal{M}(R)$, i.e., B is an affine mapping from the convex set of states into the family of probability measures on the space in which the observable takes values. It is also possible to take other measurable spaces, e.g., R^n, $\mathcal{B}(R^n)$, etc. This approach has been widely adopted, e.g., by Beltrametti and Bugajski (1995a, b, 1996, nd).

For any $s \in \mathcal{S}$ and any $M \in \mathcal{B}(R)$, we have a real number $(Bs)(M) \in [0, 1]$. Hence, we get an affine function $B_M(s) := (Bs)(M)$, $s \in \mathcal{S}$, which is an effect on $\mathcal{S}$, i.e., $B_M \in \mathcal{E}(\mathcal{S})$, where $\mathcal{E}(\mathcal{S})$ has been introduced in Section 5.

If x: $\mathcal{B}(R) \to L$, where L is any σ-effect algebra with a nonempty, convex set of σ-additive states, then the mapping

$$B_x: \quad s \mapsto s_x, \quad s \in \mathcal{S}$$

is a generalized observable. Therefore, any observable generates a generalized observable. If $\mathcal{S} = \mathrm{Tr}_1(H)$ and $L = \mathscr{E}(H)$, then an observable (= POV-measure) and a generalized observable coincide.

According to Beltrametti and Bugajski (1995b), we say that the *spectrum* of a generalized observable B, denoted by $\sigma(B)$, can be defined as the least closed set C such that $B_C = 1_{\mathcal{S}}$. Similarly, B is concentrated on $M \in \mathscr{B}(R)$ if $B_M = 1_{\mathcal{S}}$.

If B is a generalized observable and g is any Borel function on R, then by $g(B)$ we define a generalized observable $g(B)$: $s \mapsto (Bs) \circ g^{-1}$, i.e., $(g(B)s)(M) = (Bs)(g^{-1}(M))$, $M \in \mathscr{B}(R)$. If $g_n(t) = t^n$, $t \in R$, we define $B^n := g_n(B)$, $n \geq 1$.

The *mean value* of B in the state $s \in \mathcal{S}$, written $s(B)$, is defined as

$$s(B) := \int_R t\, d((Bs)(t))$$

provided that the integral on the right-hand side of the former equality exists and is finite. Similarly we have

$$s(B^n) := \int_R t\, d((B^n s)(t)) = \int_R t^n\, d((Bs)(t))$$

Let now B be a generalized observable concentrated on the interval [0, 1]. For any $n \geq 0$, we define an effect a_n on $\mathcal{S}$ via

$$a_n(s) = \int_{[0,1]} t^n\, d((Bs)(t)), \qquad s \in \mathcal{S} \tag{16}$$

Then the sequence $\{a_n\}_{n=0}^{\infty}$ is completely monotone and any a_n is an affine mapping on $\mathcal{S}$, so that $a_n \in \mathscr{E}(\mathcal{S})$ for any $n \geq 0$.

We now present a solution to a moment problem for generalized observables.

Theorem 6.1. Let $\mathcal{S}$ be a convex, nonvoid set, and let $\{a_n\}_{n=0}^{\infty}$ be a completely monotone sequence of effects on $\mathcal{S}$ with $a_0(s) = 1$ for any $s \in \mathcal{S}$. Then there is a unique generalized observable B: $\mathcal{S} \to \mathcal{M}(R)$ such that (16) holds. Equivalently, there is a convex effect algebra $L(\mathcal{S})$ with a convex, order-determining system of σ-additive states $\overline{\mathcal{S}}$ which is affinely isomorphic to $\mathcal{S}$ and a unique observable x: $\mathscr{B}(R) \to L(\mathcal{S})$ concentrated on [0, 1] such that

$$a_n(s) = \int_{[0,1]} t^n\, d\bar{s}_x(t), \qquad n \geq 0, \qquad s \in \mathcal{S} \tag{17}$$

where $\bar{s}$ is a unique σ-additive state on $L(\mathcal{S})$ corresponding in the affine ismorphism to s.

Proof. (i) Fix an element $s \in \mathcal{S}$. Then the sequence of real numbers $\{a_n(s)\}_{n=0}^{\infty}$ is completely monotone, and by a classical result of Hausdorff (1923), there is a unique probability measure p_s on $\mathcal{B}_1$ such that

$$a_n(s) = \int_{[0,1]} t^n \, dp_s(t), \qquad n \geq 0 \tag{18}$$

Define a mapping $B: \mathcal{S} \to \mathcal{M}(R)$ via

$$(Bs)(M) = p_s(M \cap [0, 1]), \qquad M \in \mathcal{B}(R) \tag{19}$$

Since any a_n is an affine functional on $\mathcal{S}$, we see that B defined via (19) is a generalized observable concentrated on [0, 1] satisfying (16).

If B_1 is another generalized observable concentrated on [0, 1] and satisfying (16), then the uniqueness of p_s on $\mathcal{B}_1$ in (18) gives $B = B_1$.

(ii) Define $L(\mathcal{S}) := \mathscr{E}(\mathcal{S})$, where $\mathscr{E}(\mathcal{S})$ is the set all effects on $\mathcal{S}$. Then $L(\mathcal{S})$ is a convex effect algebra with an order-determining system of σ-additive states $\overline{\mathcal{S}} = \{\bar{s}: s \in \mathcal{S}\}$, where $\bar{s}(f) := f(s)$, $f \in L(\mathcal{S})$, and the mapping $s \mapsto \bar{s}$ is an affine isomorphism between $\mathcal{S}$ and $\overline{\mathcal{S}}$.

Let now $B: \mathcal{S} \to \mathcal{M}(R)$ be a generalized observable from the first part of the present proof. Given $M \in \mathcal{B}(R)$, B_M: $s \mapsto (Bs)(M)$, $s \in \mathcal{S}$, gives an effect from $L(\mathcal{S})$ for which we have (a) $B_\varnothing = 0_{\mathcal{S}}$; (b) $B_R = 1_{\mathcal{S}}$; and (c) $B_{(\cup_i M_i)} = \oplus_{i=1}^{\infty} B_{M_i}$ whenever $\{M_i\}_i$ is a sequence of disjoint Borel sets from $\mathcal{B}(R)$. In other words, we have proved that the mapping x: $M \mapsto B_M$ is the observable in question for which (17) holds.

The uniqueness of x follows easily from the uniqueness of B. QED

Remark 6.2. As a by-product, we have proved in the former theorem that given a generalized observable B on the physical system with a convex, order-determining system of σ-additive states $\mathcal{S} \neq \varnothing$, we can always find a complete effect algebra $L(\mathcal{S})$ with affinely isomorphic system of completely additive states which is order determining, and an observable $x: \mathcal{B}(R) \to L(\mathcal{S})$ such that

$$Bs = s_x, \qquad s \in \mathcal{S}$$

i.e.,

$$(Bs)(M) = s(x(M)), \qquad M \in \mathcal{B}(R)$$

This effect algebra is a convex effect algebra $\mathscr{E}(\mathcal{S})$ defined in Section 5 which consists of all affine fuzzy sets on $\mathcal{S}$.

Remark 6.3. Beltrametti and Bugajski (n.d., Theorem 1) showed a similar relation between generalized observables and observables to that in Remark 6.2.

ACKNOWLEDGMENTS

One of the authors (A.D.) is grateful to the Mathematics Department of the University "Federico II" of Naples for their hospitality during his stay in October 1996, when the present paper was completed. A.D. also thanks Prof. S. Bugajski for stimulating discussions which were inspiring for solving the problem under question. This paper was partially supported by Grants G 228/94 and G 229/94 SAV, Bratislava, Slovakia, and by the Ministero dell'Università e della Ricerca Scientifica e Tecnologica, Italy.

REFERENCES

Beltrametti, E. G., and Bugajski, S. (1995a). Quantum observables in classical frameworks, *International Journal of Theoretical Physics*, **34**, 1221–1229.

Beltrametti, E. G., and Bugajski, S. (1995b). A classical extension of quantum mechanics, *Journal of Physics A: Mathematical and General*, **28**, 3329–3343.

Beltrametti, E. G., and Bugajski, S. (1996). The Bell phenomenon in classical frameworks, *Journal of Physics A: Mathematical and General*, **29**, 247–261.

Beltrametti, E. G., and Bugajski, S. (n.d.). Effects algebras and statistical physical theories, submitted.

Busch, P., Lahti, P. J., and Mittelstaedt, P. (1991). *The Quantum Theory of Measurement*, Springer-Verlag, Berlin.

Dvurečenskij, A. (1993). *Gleason's Theorem and its Applications*, Kluwer, Dordrecht, and Ister Science Press, Bratislava.

Dvurečenskij, A. (n.d.). Fuzzy set representation of some quantum structures, *Fuzzy Sets and Systems*, to appear.

Foulis, D. J., and Bennett, M. K. (1993). Tensor products of orthoalgebras, *Order*, **10**, 271–282.

Giuntini, R., and Greuling, H. (1989). Toward a formal language for unsharp properties, *Foundations of Physics*, **19**, 931–945.

Hausdorff, F. (1921a). Summationsmethoden und Momentfolgen, I, *Mathematische Zeitschrift*, **9**, 74–109.

Hausdorff, F. (1921b). Summationsmethoden und Momentfolgen, II, *Mathematische Zeitschrift*, **9**, 280–299.

Hausdorff, F. (1923). Momentprobleme für ein endliches Interval, *Mathematische Zeitschrift*, **16**, 220–248.

Kadison, R. V. (1952). A generalized Schwarz inequality and algebraic invariants for operators and algebras, *Annals of Mathematics*, **56**, 494–503.

Kôpka, F., and Chovanec, F. (1994). D-posets, *Mathematica Slovaca*, **44**, 21–34.

Reed, M., and Simon, B. (1972). *Methods of Modern Mathematical Physics I, Functional Analysis*, Academic Press, New York.

Riesz, F., and Sz.-Nagy, B. (1955). *Leçons d'Analyse Fonctionelle*, 3rd ed., Akadémiai Kiadó, Budapest.

Shohat, J., and Tamarkin, J. (1943). *The Problem of Moments*, American Mathematical Society, Providence, Rhode Island.

Widder, D. V. (1946). *Laplace Transform*, Princeton University Press, Princeton, New Jersey.